Lesson 31 - Lagrange Multipliers
  I. Picture
 II Technique
 III. Examples

I. Picture

We want to find maximum area that can be enclosed
by a rectangular fence that has perimeter of 100 ft.

$$\text{Max} \quad A = xy$$
$$\text{Subject to:} \quad 2x + 2y = 100$$
$$(S.t.)$$
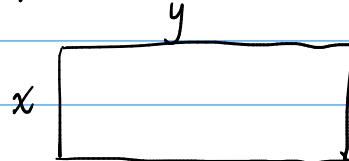
Calc I techniques can solve this

Because $A(x,y) = xy$ is a fctn of 2 vars, we
have new techniques for solving thi

max $\quad\quad A(x, y) = xy$ ← objective function

Subject to: $\quad 2x + 2y = 100$ ← constraint equation

Pictures from Geogebra:

where you attain max A value →

← where you attain max A value
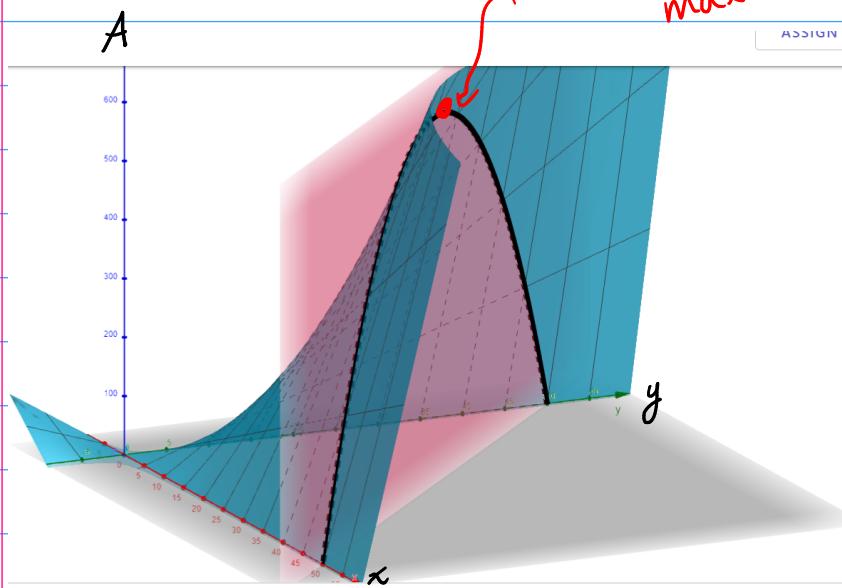
Blue: $A = xy$
pink: $2x + 2y = 100$

See gradient file on our website

One way to encode that two functions have the same tan line/
normal lines @ (a, b) is to say that

$$f_x(a, b) = \lambda g_x(a, b)$$
$$\text{and} \quad f_y(a, b) = \lambda g_y(a, b)$$

for some scalar (constant) λ

II. Technique.

← objective function

To solve $\quad\quad$ max or min $\quad z = f(x, y)$

$\quad\quad\quad\quad\quad$ subject to $\quad\quad g(x, y) = 0$

← constraint function set $= 0$

① Solve the system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases}$$

for (x,y) and λ

② check to make sure your max or min occurs @ (x,y) by substituting any other point on $g(x,y) = 0$ into $z = f(x,y)$.

Ⅲ. Examples

$$\text{max } A(x,y) = xy \overset{f(x,y)}{\frown}$$
$$\text{s.t. } \quad 2x + 2y = 100 \quad \longrightarrow \quad \underbrace{2x + 2y - 100}_{g(x,y)} = 0$$

$$\begin{cases} A_x(x,y) = \lambda g_x(x,y) \\ A_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases} \rightarrow \begin{cases} y = \lambda \cdot 2 & \text{①} \\ x = \lambda \cdot 2 & \text{②} \\ 2x + 2y - 100 = 0 & \text{③} \end{cases}$$

① and ② $\longrightarrow$ ③

$$2(2\lambda) + 2(2\lambda) - 100 = 0$$
$$8\lambda = 100$$
$$\lambda = 12.5$$

$\lambda = 12.5 \rightarrow$ ① and ②
$$y = (12.5) \cdot 2 = 25$$
$$x = (12.5) \cdot 2 = 25$$

Let's hope max is @ $(x,y) = (25, 25)$
$$A(25, 25) = (25)(25) = 625$$

Pick any other point on $2x + 2y - 100 = 0$
$$2x + 2y = 100$$

Often easiest to pick a
pt where $x = 0$ or $y = 0$
I choose $x = 0$ $2(0) + 2y = 100$
$$y = 50$$

$(0, 50) \rightarrow A(0, 50) = 0 \cdot 50 = 0 < 625$
So max area is 625 ft^2 when the fence is a 25×25 square.

EX (Openstax, Calc III, Ex 4.4.2)

min $f(x, y) = x^2 + 4y^2 - 2x + 8y$
s.t. $x + 2y = 7$
$$g(x, y) = x + 2y - 7 = 0$$

$$\begin{cases} f_x(x, y) = \lambda\, g_x(x, y) \\ f_y(x, y) = \lambda\, g_y(x, y) \\ g(x, y) = 0 \end{cases} \rightarrow \begin{cases} 2x - 2 = \lambda \cdot 1 & ① \\ 8y + 8 = \lambda \cdot -2 & ② \\ x + 2y = 7 & ③ \end{cases}$$

① $\lambda = 2x - 2 \longrightarrow$ ② $8y + 8 = (2x - 2) \cdot 2$
$$8y + 8 = 4x - 4$$
$$4x = 8y + 12$$
$$x = 2y + 3 \quad ④$$
$$\downarrow ③$$

$$2y + 3 + 2y = 7$$
$$4y = 4$$
$$y = 1 \longrightarrow ④ \quad x = 2(1) + 3 = 5$$

$(5, 1)$ $f(5, 1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27$
Any other point on $x + 2y = 7$

I choose $y=0 \Rightarrow x+0=7 \Rightarrow x=7$

$(7,0)$ $f(7,0) = 7^2 + 4(0)^2 - 2(7) + 8(0) = 35 > 27$

Minimum of 27 @ $(5,1)$

[EX] (openstax, Calc III, §4.8 #358)

Find max and min of
$$f(x,y) = x^2 y$$
subject to $\underbrace{x^2 + 2y^2 - 6}_{g(x,y)} = 0$

$$\begin{cases} 2xy = \lambda 2x & \text{①} \\ x^2 = \lambda 4y & \text{②} \\ x^2 + 2y^2 = 6 & \text{③} \end{cases}$$

Let's look @ ①.
$$2xy = \lambda 2x$$
$$2xy - 2x\lambda = 0$$
$$2x(y - \lambda) = 0$$

You cannot divide by x.

$x = 0$ or $y = \lambda$

Case 1: $x = 0$
$\rightarrow 3$
$0^2 + 2y^2 = 6$
$y^2 = 3$
$y = \pm\sqrt{3}$

Points to consider
$(0, \sqrt{3})$
$(0, -\sqrt{3})$

Case 2: $y = \lambda$

② $x^2 = \lambda 4y$
$\Rightarrow x^2 = y \, 4y$
$x^2 = 4y^2$

③ $x^2 + 2y^2 = 6$
$4y^2 + 2y^2 = 6$
$6y^2 = 6$
$y^2 = 1$
$y = \pm 1$

$\rightarrow$③   points $(2,1) \; (2,-1)$
$(-2,1) \; (-2,-1)$